

# BIHARMONIC HYPERSURFACES IN A RIEMANNIAN MANIFOLD WITH NON-POSITIVE RICCI CURVATURE

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**ABSTRACT.** In this paper, we show that, for a biharmonic hypersurface  $(M, g)$  of a Riemannian manifold  $(N, h)$  of non-positive Ricci curvature, if  $\int_M |H|^2 v_g < \infty$ , where  $H$  is the mean curvature of  $(M, g)$  in  $(N, h)$ , then  $(M, g)$  is minimal in  $(N, h)$ . Thus, for a counter example  $(M, g)$  in the case of hypersurfaces to the generalized Chen's conjecture (cf. Sect.1), it holds that  $\int_M |H|^2 v_g = \infty$ .

## 1. INTRODUCTION AND STATEMENT OF RESULTS.

In this paper, we consider an isometric immersion  $\varphi : (M, g) \rightarrow (N, h)$ , of a Riemannian manifold  $(M, g)$  of dimension  $m$ , into another Riemannian manifold  $(N, h)$  of dimension  $n = m + 1$ . We have

$$\nabla_{\varphi_* X}^N \varphi_* Y = \varphi_*(\nabla_X Y) + k(X, Y)\xi,$$

for vector fields  $X$  and  $Y$  on  $M$ , where  $\nabla, \nabla^N$  are the Levi-Civita connections of  $(M, g)$  and  $(N, h)$ , respectively,  $\xi$  is the unit normal vector field along  $\varphi$ , and  $k$  is the second fundamental form. Let  $A : T_x M \rightarrow T_x M$  ( $x \in M$ ) be the shape operator defined by  $g(AX, Y) = k(X, Y)$ , ( $X, Y \in T_x M$ ), and  $H$ , the mean curvature defined by  $H := \frac{1}{m} \text{Tr}_g(A)$ . Then, let us recall the following **B.Y. Chen's conjecture** (cf. [3], [4]):

*Let  $\varphi : (M, g) \rightarrow (\mathbb{R}^n, g_0)$  be an isometric immersion into the standard Euclidean space. If  $\varphi$  is biharmonic (see Sect. 2), then, it is minimal.*

This conjecture is still open up to now, and let us recall also the following **generalized B.Y. Chen's conjecture** (cf. [3], [2]):

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Let  $\varphi : (M, g) \rightarrow (N, h)$  be an isometric immersion, and the sectional curvature of  $(N, h)$  is non-positive. If  $\varphi$  is biharmonic, then, it is minimal.

Oniciuc ([9]) and Ou ([11]) showed this is true if  $H$  is constant.

In this paper, we show

**Theorem 1.1.** *Assume that  $(M, g)$  is complete and the Ricci tensor  $\text{Ric}^N$  of  $(N, h)$  satisfies that*

$$\text{Ric}^N(\xi, \xi) \leq |A|^2. \quad (1.1)$$

*If  $\varphi : (M, g) \rightarrow (N, h)$  is biharmonic (cf. Sect. 2) and satisfies that*

$$\int_M H^2 v_g < \infty, \quad (1.2)$$

*then,  $\varphi$  has constant mean curvature, i.e.,  $H$  is constant.*

As a direct corollary, we have

**Corollary 1.2.** *Assume that  $(M, g)$  is a complete Riemannian manifold of dimension  $m$  and  $(N, h)$  is a Riemannian manifold of dimension  $m+1$  whose Ricci curvature is non-positive. If an isometric immersion  $\varphi : (M, g) \rightarrow (N, h)$  is biharmonic and satisfies that  $\int_M H^2 v_g < \infty$ , then,  $\varphi$  is minimal.*

By our Corollary 1.2, if there would exist a counter example (cf. [11]) in the case  $\dim N = \dim M + 1$ , then it must hold that

$$\int_M H^2 v_g = \infty, \quad (1.3)$$

which imposes the strong condition on the behaviour of the boundary of  $M$  at infinity. Indeed, (1.3) implies that either  $H$  is unbounded on  $M$ , or it holds that  $H^2 \geq C$  on an open subset  $\Omega$  of  $M$  with infinite volume, for some constant  $C > 0$ .

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## 2. PRELIMINARIES.

In this section, we prepare general materials about harmonic maps and biharmonic maps of a complete Riemannian manifold into another Riemannian manifold (cf. [5]).

Let  $(M, g)$  be an  $m$ -dimensional complete Riemannian manifold, and the target space  $(N, h)$  is an  $n$ -dimensional Riemannian manifold. For every  $C^\infty$  map  $\varphi$  of  $M$  into  $N$ , and relatively compact domain  $\Omega$  in  $M$ , the *energy functional* on the space  $C^\infty(M, N)$  of all  $C^\infty$  maps of  $M$  into  $N$  is defined by

$$E_\Omega(\varphi) = \frac{1}{2} \int_\Omega |d\varphi|^2 v_g,$$

and for a  $C^\infty$  one parameter deformation  $\varphi_t \in C^\infty(M, N)$  ( $-\epsilon < t < \epsilon$ ) of  $\varphi$  with  $\varphi_0 = \varphi$ , the variation vector field  $V$  along  $\varphi$  is defined by  $V = \frac{d}{dt} \Big|_{t=0} \varphi_t$ . Let  $\Gamma_\Omega(\varphi^{-1}TN)$  be the space of  $C^\infty$  sections of the induced bundle  $\varphi^{-1}TN$  of the tangent bundle  $TN$  by  $\varphi$  whose supports are contained in  $\Omega$ . For  $V \in \Gamma_\Omega(\varphi^{-1}TN)$  and its one-parameter deformation  $\varphi_t$ , the *first variation formula* is given by

$$\frac{d}{dt} \Big|_{t=0} E_\Omega(\varphi_t) = - \int_\Omega \langle \tau(\varphi), V \rangle v_g.$$

The *tension field*  $\tau(\varphi)$  is defined globally on  $M$  by

$$\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i), \quad (2.1)$$

where

$$B(\varphi)(X, Y) = \nabla_{\varphi_*(X)}^N \varphi_*(Y) - \varphi_*(\nabla_X Y)$$

for  $X, Y \in \mathfrak{X}(M)$ . Then, a  $C^\infty$  map  $\varphi : (M, g) \rightarrow (N, h)$  is *harmonic* if  $\tau(\varphi) = 0$ . For a harmonic map  $\varphi : (M, g) \rightarrow (N, h)$ , the *second variation formula* of the energy functional  $E_\Omega(\varphi)$  is

$$\frac{d^2}{dt^2} \Big|_{t=0} E_\Omega(\varphi_t) = \int_\Omega \langle J(V), V \rangle v_g$$

where

$$J(V) := \overline{\Delta}V - \mathcal{R}(V),$$

$$\overline{\Delta}V := \overline{\nabla}^* \overline{\nabla}V = - \sum_{i=1}^m \{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \},$$

$$\mathcal{R}(V) := \sum_{i=1}^m R^N(V, \varphi_*(e_i)) \varphi_*(e_i).$$

Here,  $\bar{\nabla}$  is the induced connection on the induced bundle  $\varphi^{-1}TN$ , and  $R^N$  is the curvature tensor of  $(N, h)$  given by  $R^N(U, V)W = [\nabla_U^N, \nabla_V^N]W - \nabla_{[U, V]}^N W$  ( $U, V, W \in \mathfrak{X}(N)$ ).

The *bienergy functional* is defined by

$$E_{2,\Omega}(\varphi) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2 v_g,$$

and the *first variation formula* of the bienergy is given (cf. [7]) by

$$\left. \frac{d}{dt} \right|_{t=0} E_{2,\Omega}(\varphi_t) = - \int_{\Omega} \langle \tau_2(\varphi), V \rangle v_g$$

where the *bitension field*  $\tau_2(\varphi)$  is defined globally on  $M$  by

$$\tau_2(\varphi) = J(\tau(\varphi)) = \bar{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \quad (2.2)$$

and a  $C^\infty$  map  $\varphi : (M, g) \rightarrow (N, h)$  is called to be *biharmonic* if

$$\tau_2(\varphi) = 0. \quad (2.3)$$

### 3. SOME LEMMA FOR THE SCHRÖDINGER TYPE EQUATION

In this section, we prepare some simple lemma of the Schrödinger type equation of the Laplacian  $\Delta_g$  on an  $m$ -dimensional non-compact complete Riemannian manifold  $(M, g)$  defined by

$$\Delta_g f := \sum_{i=1}^m e_i(e_i f) - \nabla_{e_i} e_i f \quad (f \in C^\infty(M)), \quad (3.1)$$

where  $\{e_i\}_{i=1}^m$  is a locally defined orthonormal frame field on  $(M, g)$ .

**Lemma 3.1.** *Assume that  $(M, g)$  is a complete non-compact Riemannian manifold, and  $L$  is a non-negative smooth function on  $M$ . Then, every smooth  $L^2$  function  $f$  on  $M$  satisfying the Schrödinger type equation*

$$\Delta_g f = L f \quad (\text{on } M) \quad (3.2)$$

*must be a constant.*

*Proof.* Take any point  $x_0$  in  $M$ , and for every  $r > 0$ , let us consider the following cut-off function  $\eta$  on  $M$ :

$$\begin{cases} 0 \leq \eta(x) \leq 1 & (x \in M), \\ \eta(x) = 1 & (x \in B_r(x_0)), \\ \eta(x) = 0 & (x \notin B_{2r}(x_0)), \\ |\nabla \eta| \leq \frac{2}{r} & (\text{on } M), \end{cases} \quad (3.3)$$

where  $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$ , and  $d$  is the distance of  $(M, g)$ . Multiply  $\eta^2 f$  on (3.2), and integrate it over  $M$ , we have

$$\int_M (\eta^2 f) \Delta_g f v_g = \int_M L \eta^2 f^2 v_g. \quad (3.4)$$

By the integration by part for the left hand side, we have

$$\int_M (\eta^2 f) \Delta_g f v_g = - \int_M g(\nabla(\eta^2 f), \nabla f) v_g. \quad (3.5)$$

Here, we have

$$\begin{aligned} g(\nabla(\eta^2 f), \nabla f) &= 2\eta f g(\nabla\eta, \nabla f) + \eta^2 g(\nabla f, \nabla f) \\ &= 2\eta f \langle \nabla\eta, \nabla f \rangle + \eta^2 |\nabla f|^2, \end{aligned} \quad (3.6)$$

where we use  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  instead of  $g(\cdot, \cdot)$  and  $g(u, u) = |u|^2$  ( $u \in T_x M$ ), for simplicity. Substitute (3.6) into (3.5), the right hand side of (3.5) is equal to

$$\begin{aligned} RHS \text{ of (3.5)} &= - \int_M 2\eta f \langle \nabla\eta, \nabla f \rangle v_g - \int_M \eta^2 |\nabla f|^2 v_g \\ &= -2 \int_M \langle f \nabla\eta, \eta \nabla f \rangle v_g - \int_M \eta^2 |\nabla f|^2 v_g. \end{aligned} \quad (3.7)$$

Here, applying Young's inequality: for every  $\epsilon > 0$ , and every vectors  $X$  and  $Y$  at each point of  $M$ ,

$$\pm 2 \langle X, Y \rangle \leq \epsilon |X|^2 + \frac{1}{\epsilon} |Y|^2, \quad (3.8)$$

to the first term of (3.7), we have

$$\begin{aligned} RHS \text{ of (3.7)} &\leq \epsilon \int_M |\eta \nabla f|^2 v_g + \frac{1}{\epsilon} \int_M |f \nabla\eta|^2 v_g - \int_M \eta^2 |\nabla f|^2 v_g \\ &= -(1 - \epsilon) \int_M \eta^2 |\nabla f|^2 v_g + \frac{1}{\epsilon} \int_M f^2 |\nabla\eta|^2 v_g. \end{aligned} \quad (3.9)$$

Thus, by (3.5) and (3.9), we obtain

$$\int_M L \eta^2 f^2 v_g + (1 - \epsilon) \int_M \eta^2 |\nabla f|^2 v_g \leq \frac{1}{\epsilon} \int_M f^2 |\nabla\eta|^2 v_g. \quad (3.10)$$

Now, putting  $\epsilon = \frac{1}{2}$ , (3.10) implies that

$$\int_M L \eta^2 f^2 v_g + \frac{1}{2} \int_M \eta^2 |\nabla f|^2 v_g \leq 2 \int_M f^2 |\nabla\eta|^2 v_g. \quad (3.11)$$

Since  $\eta = 1$  on  $B_r(x_0)$  and  $|\nabla\eta| \leq \frac{2}{r}$ , and  $L \geq 0$  on  $M$ , we have

$$0 \leq \int_{B_r(x_0)} L f^2 v_g + \frac{1}{2} \int_{B_r(x_0)} |\nabla f|^2 v_g \leq \frac{8}{r^2} \int_M f^2 v_g. \quad (3.12)$$

Since  $(M, g)$  is non-compact and complete,  $r$  can tend to infinity, and  $B_r(x_0)$  goes to  $M$ . Then we have

$$0 \leq \int_M L f^2 v_g + \frac{1}{2} \int_M |\nabla f|^2 v_g \leq 0 \quad (3.13)$$

since  $\int_M f^2 v_g < \infty$ . Thus, we have  $L f^2 = 0$  and  $|\nabla f| = 0$  (on  $M$ ) which implies that  $f$  is a constant.  $\square$

#### 4. BIHARMONIC ISOMETRIC IMMERSIONS.

In this section, we consider a hypersurface  $M$  of an  $(m+1)$ -dimensional Riemannian manifold  $(N, h)$ . Recently, Y-L. Ou showed (cf. [10])

**Theorem 4.1.** *Let  $\varphi : (M, g) \rightarrow (N, h)$  be an isometric immersion of an  $m$ -dimensional Riemannian manifold  $(M, g)$  into another  $(m+1)$ -dimensional Riemannian manifold  $(N, h)$  with the mean curvature vector field  $\eta = H\xi$ , where  $\xi$  is the unit normal vector field along  $\varphi$ . Then,  $\varphi$  is biharmonic if and only if the following equations hold:*

$$\begin{cases} \Delta_g H - H |A|^2 + H \operatorname{Ric}^N(\xi, \xi) = 0, \\ 2A(\nabla H) + \frac{m}{2} \nabla(H^2) - 2H(\operatorname{Ric}^N(\xi))^T = 0, \end{cases} \quad (4.1)$$

where  $\operatorname{Ric}^N : T_y N \rightarrow T_y N$  is the Ricci transform which is defined by  $h(\operatorname{Ric}^N(Z), W) = \operatorname{Ric}^N(Z, W)$  ( $Z, W \in T_y N$ ),  $(\cdot)^T$  is the tangential component corresponding to the decomposition of  $T_{\varphi(x)} N = \varphi_*(T_x M) \oplus \mathbb{R}\xi_x$  ( $x \in M$ ), and  $\nabla f$  is the gradient vector field of  $f \in C^\infty(M)$  on  $(M, g)$ , respectively.

Due to Theorem 4.1 and Lemma 3.1, we can show immediately our Theorem 1.1.

(Proof of Theorem 1.1.)

Let us denote by  $L := |A|^2 - \operatorname{Ric}^N(\xi, \xi)$  which is a smooth non-negative function on  $M$  due to our assumption. Then, the first equation is reduced to the following Schrödinger type equation:

$$\Delta_g f = L f, \quad (4.2)$$

where  $f := H$  is a smooth  $L^2$  function on  $M$  by the assumption (1.2).

Assume that  $M$  is compact. In this case, by (4.2) and the integration by part, we have

$$0 \leq \int_M L f^2 v_g = \int_M f (\Delta_g f) v_g = - \int_M g(\nabla f, \nabla f) v_g \leq 0, \quad (4.3)$$

which implies that  $\int_M g(\nabla f, \nabla f) v_g = 0$ , that is,  $f$  is constant.

Assume that  $M$  is non-compact. In this case, we can apply Lemma 3.1 to (4.2). Then, we have that  $f = H$  is a constant.  $\square$

(Proof of Corollary 1.2.)

Assume that  $\text{Ric}^N$  is non-positive. Since  $L = |A|^2 - \text{Ric}^N(\xi, \xi)$  is non-negative,  $H$  is constant due to Theorem 1.1. Then, due to (4.1), we have that  $H L = 0$  and  $H (\text{Ric}^N(\xi))^T = 0$ . If  $H \neq 0$ , then  $L = 0$ , i.e.,

$$\text{Ric}^N(\xi, \xi) = |A|^2. \quad (4.4)$$

By our assumption,  $\text{Ric}^N(\xi, \xi) \leq 0$ , and the right hand side of (4.4) is non-negative, so we have  $|A|^2 = 0$ , i.e.,  $A \equiv 0$ . This contradicts  $H \neq 0$ . We have  $H = 0$ .  $\square$

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